

Kerr AdS and Dyonic black holes as Heat Engine

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ABSTRACT: As we know, the cosmological constant in different theories of gravity acts as a thermodynamics variable. The cosmological constant exists in different actions of gravity and also appears in the solution of such theories. These lead to use the black hole as a heat engines. Also, there are two values for the cosmological constant as positive and negative values. The case of negative cosmological constant supplies a natural realization of these engines in terms of the field theory description of the fluids to which they are holographically dual. In this paper, we are going to define heat engines for two different black holes as Dyonic BH and Kerr AdS BH. And also, we calculate maximum efficiency for these two black holes.

Contents

| | | |
|----------|--|-----------|
| 1 | Introduction | 1 |
| 2 | Thermodynamic cycles and Heat Engines | 2 |
| 3 | Dyonic black hole solution | 4 |
| 4 | Kerr Ads black hole solution | 7 |
| 5 | Conclusion | 10 |

1 Introduction

Thermodynamics of black holes was achieved by Hawking, Carter and Bardeen for the first time. When Hawking radiation discovered, they believed black holes are thermodynamics systems [1] and their temperature is related to event horizon. It means that dynamic laws of black holes with thermodynamic laws are similar to each other [2].

Four mechanic laws of black holes were formulated by Hawking, Carter and Bardeen in 1973 [3] which are similar to thermodynamic laws, and also Chistodoulou confirmed theses [4].

The classic subject of black hole thermodynamics [5–8], which relates the mass M , surface gravity κ and area A of a black hole with the energy U , temperature T , and entropy S , as follows,

$$M = U \quad , \quad T = \frac{\kappa}{2\pi} \quad , \quad S = \frac{A}{4}. \quad (1.1)$$

And this subject has been extended to include black hole counterparts for the pressure P and volume V . Ref.s. [9–20]. The cosmological constant of space time relates to pressure as $P = -\frac{\Lambda}{8\pi}$ and thermodynamic volume of black holes is defined as: $V = (\frac{\partial M}{\partial P})_{S, \phi_i, J_k}$ (here we are using geometric units $G_N = \hbar = c = k = 1$). The black holes may have other parameters such as gauge charge q_i and angular momentum J_j with their conjugate, as potential ϕ_i and angular velocity Ω_j .

Recently, the variation of cosmological constant Λ , in the first law of black holes thermodynamics has been attention by Ref.s. [21–23]. So, pressure is a thermodynamic variable and mass will be enthalpy, it is given by,

$$M = H \equiv U + PV. \quad (1.2)$$

The first law of black holes thermodynamics in an extended (including P and V variables) phase space in four dimensions with an electric charge and rotation is,

$$dM = TdS + VdP + \Phi dq + \Omega dJ. \quad (1.3)$$

When P is treated as constant (cosmological constant is not allowed to vary), above relation reduces to the standard first law in the 'non-extended' phase space. The thermodynamic volume of static black holes is equal to volume of horizon radius. For example in 4-dimensional Reissner- Nordstrom and Schwarzschild black holes we have following equation,

$$V = \frac{4}{3}\pi r_H^3. \quad (1.4)$$

Also, the entropy is related to horizon radius, then volume and entropy are related to each other in static black holes. Thermodynamics of black holes not only determines standard thermodynamic variables such as temperature and entropy, but also has extensive phase structure in analogy with known non gravitational thermodynamic systems and admit critical phenomena.

Historically, the study of thermodynamic properties of AdS black hole was started with impressive article of Hawking and Page [24], they have shown that a phase transition in phase space of AdS Schwarzschild black hole (non rotating uncharged). Then our understanding about phase transition and critical phenomena of systems with more complicated background increased, [25, 26]. The critical behavior of RN-AdS black hole in non extended phase space ($\Lambda=\text{cte}$ thus $P=\text{cte}$), at canonical ensemble (fixed charge) was studied in [27, 28] that has been observed phase transition behavior is similar to Liquid-gas phase transition. Also, the critical behavior of this black hole in extended phase space is remarkable coincidence with Van der Waals fluid studied by [29].

In Ref. [30], authors considered a Dyonic black hole in (3+1) dimensions (this black hole has a magnetic charge in addition to electric charge), and they have shown that putting black hole into ensemble of fixed electric charge and magnetic charge results is similar to [27–29] except that there we have $q_E^2 \rightarrow q_E^2 + q_M^2$.

After studying of phase transition, it is interesting, whether we can define classical cycles for black holes like usual thermodynamic systems? It means that after phase transition of small black hole to large black hole, again the (LBH) can reduce to (SBH), in other words, system can back to primary state.

The holographic heat engine has been studied in Ref [31] and has examined for charge AdS black hole. In this paper, we are going to consider for Dyonic BH and Kerr AdS BH. First of all we review thermodynamic cycles and heat engine and then we study thermodynamic cycles for Dyonic BH and Kerr AdS BH. Also, here we compare two black holes in cycle point of view and obtain some new results.

2 Thermodynamic cycles and Heat Engines

By using volume, pressure, temperature and entropy we can calculate heat energy and mechanical useful work. To do this, we start with equation of state (function of $P(V,T)$) and define an engine as a close path in P - V plane which receives Q_H and gives Q_C . From the first law of thermodynamics, total mechanical work is defined as, $W = Q_H - Q_C$. Therefore, the efficiency of heat engine is, $\eta = \frac{W}{Q_H} = 1 - \frac{Q_C}{Q_H}$.

Some of the classic cycles involve a pair of isotherms at temperature T_H and T_C , where

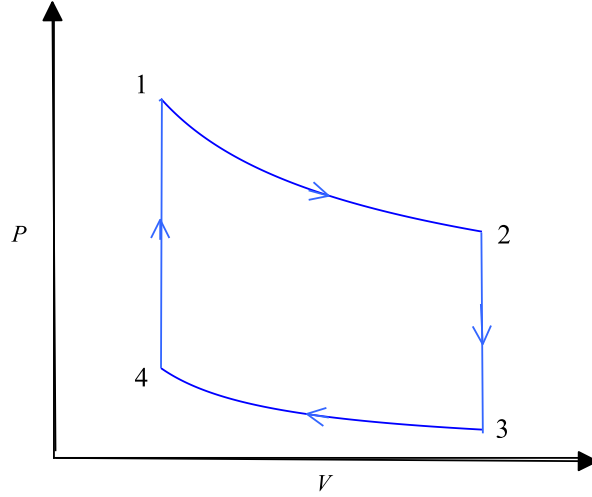


Figure 1. The Carnot engine

$T_H > T_C$, where there is an isothermal expansion while some heat is being absorbed, and an isothermal compression during expulsion of some heat. By using different method one can relate two systems each other. The first method here is isochoric path like classical Stirling cycle and the second one is adiabatic path like classical Carnot cycle.

We know that a whole heat engine is fully reversible (since the total entropy flow zero). Therefore, the engine should have Carnot efficiency ($\eta = 1 - \frac{T_C}{T_H}$) which is maximum efficiency that every heat engine can have. Any higher efficiency would violate the second Law. It is very important that how can reach to such efficiency in heat engine of black holes. So, the form of path for the definition of cycle is important. For the static black holes, the thermodynamic volume V and the entropy S are not independent. It means that adiabats and isochores are the same, in that case Carnot and Stirling coincide to each other. So, the efficiency of cycle can be calculated easily.

So along the upper isotherm, we have the following heat flow,

$$Q_H = T_H \Delta S_{1 \rightarrow 2} = T_H \left(\frac{3}{4\pi} \right)^{\frac{2}{3}} \pi (V_2^{\frac{2}{3}} - V_1^{\frac{2}{3}}). \quad (2.1)$$

And also along the lower, isotherm the heat flow will be as,

$$Q_C = T_C \Delta S_{3 \rightarrow 4} = T_C \left(\frac{3}{4\pi} \right)^{\frac{2}{3}} \pi (V_3^{\frac{2}{3}} - V_4^{\frac{2}{3}}). \quad (2.2)$$

Since $V_1 = V_4$ and $V_2 = V_3$, the efficiency becomes,

$$\eta = 1 - \frac{T_C}{T_H}. \quad (2.3)$$

3 Dyonic black hole solution

Now we are going to consider the Dyonic black hole system. The solution of dyonic-dilaton AdS black hole can be used for maximal gauge super gravity in 4-dimension [32]. In that case the corresponding action will be as,

$$I = \frac{1}{16\pi G_4} \int d^4x \sqrt{g} (-R + F^2 - \frac{6}{b^2}). \quad (3.1)$$

The equation of motion is given by,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \frac{3}{b^2}g_{\mu\nu} = 2(F_{\mu\lambda}F_{\nu}^{\lambda} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta});$$

$$\nabla_{\mu}F^{\mu\nu} = 0. \quad (3.2)$$

One can write a static spherically symmetric solution to this equation as,

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2, \quad (3.3)$$

where

$$f(r) = (1 + \frac{r^2}{b^2} - \frac{2M}{r} + \frac{q_E^2 + q_M^2}{r^2}). \quad (3.4)$$

The electromagnetic four-potential A_{μ} is,

$$A = (-\frac{q_E}{r} + \frac{q_E}{r_+})dt + (q_M \cos\theta)d\phi, \quad (3.5)$$

where q_E , q_M and M are electric charge, magnetic charge and mass of black hole respectively. The horizon of black hole is given by,

$$f(r_+) = (1 + \frac{r_+^2}{b^2} - \frac{2M}{r_+} + \frac{q_E^2 + q_M^2}{r_+^2}) = 0, \quad (3.6)$$

and electric potential Φ_E is defined by following equation,

$$\Phi_E = \frac{q_E}{r_+}. \quad (3.7)$$

The Hawking temperature of this black hole is following,

$$T = \frac{1}{\beta} = \frac{1}{4\pi r_+} [1 + \frac{3r_+^2}{b^2} - \Phi_E^2 - \frac{q_M^2}{r_+^2}]. \quad (3.8)$$

The pressure is given by,

$$P = -\frac{\Lambda}{8\pi} = \frac{3}{8\pi} \frac{1}{b^2}, \quad (3.9)$$

and the thermodynamical volume is,

$$V = \frac{4\pi}{3} r_+^3. \quad (3.10)$$

We can write the temperature in equation (15) in terms of S and P as follows,

$$T = \frac{1}{4\sqrt{\pi}} \frac{1}{\sqrt{S}} (1 + 8PS - \Phi_E^2 - \frac{\pi q_M^2}{S}). \quad (3.11)$$

One can write the pressure as a function of T , r_+ , Φ_E and q_M from equation (15) as follows,

$$P = \frac{T}{v} - \frac{1 - \Phi_E^2}{2\pi v^2} + \frac{2q_M^2}{\pi v^4}, \quad (3.12)$$

where, $v = 2r_+$ can be identified with the specific volume of the system [29]. This equation describes different phases of a dyonic black hole in a fixed electric potential and magnetic charge ensemble which is similar to extended liquid-gas phase diagram.

In thermodynamic, heat capacity is an important measurable physical quantity. It determine the amount of requisite heat to change the temperature of an object by a given amount. There are two different heat capacities for a system, heat capacity at constant pressure and heat capacity at constant volume. Heat capacity can be calculated by the standard thermodynamic relations, which is given by,

$$C_V = T \left. \frac{\partial S}{\partial T} \right|_V, \quad C_P = T \left. \frac{\partial S}{\partial T} \right|_P. \quad (3.13)$$

The entropy is given by,

$$S = \frac{A_H}{4} = \pi r_+^2 = \pi \left(\frac{3V}{4\pi} \right)^{\frac{2}{3}}. \quad (3.14)$$

Thus we have $C_V = 0$ and C_P which is given by,

$$C_P = 2S \frac{(8PS^2 + S(1 - \Phi_E^2) - \pi q_M^2)}{(8PS^2 + S(1 - \Phi_E^2) + 3\pi q_M^2)}. \quad (3.15)$$

One can write P as a function of thermodynamical volume from equation (19), so we have,

$$P = \frac{T}{V^{\frac{1}{3}}} \left(\frac{\pi}{6} \right)^{\frac{1}{3}} - \frac{1 - \Phi_E^2}{2\pi V^{\frac{2}{3}}} \left(\frac{\pi}{6} \right)^{\frac{2}{3}} + \frac{2q_M^2}{\pi V^{\frac{4}{3}}} \left(\frac{\pi}{6} \right)^{\frac{4}{3}}. \quad (3.16)$$

P - V diagram is depicted for fixed (T, Φ_E, q_M) in figure (2).

In figure (2a), there are three different solutions for black hole horizon. These are identified as Branch-1 (Black), Branch-2 (Red) and Branch-3 (Green). For large P only Branch-1 (SBH) exists, while for low P Branch-3 (LBH) is the only solution. On the other hand in figure (2) Branch-2 indicates unstable phase thermodynamically. In figure (2b) we see that above a critical temperature, Branch-2 totally disappears and Branch-1 and 3 coalesce, therefore phase transition occur for $T < T_C$.

In phase transition, SBH reduces to LBH, if after that LBH again transfers to SBH, we can define classical cycle for black hole. As before we explained, the SBH absorbs amount of heat Q_H along isothermal expansion and we know also it exists at high pressure.

Actually, an explicit expression for C_P would suggest that we ought to have a new engine, involving two isobars and two isochores/adiabats as figure (3). The work done in this cycle is,

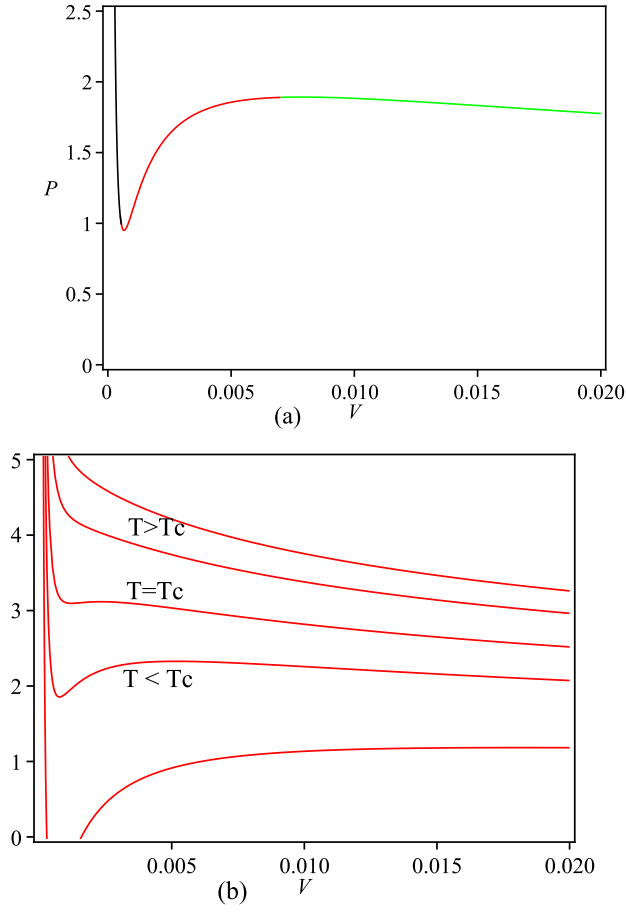


Figure 2. (a) P-V diagram for all other parameters held constant ($q_M = 0.028$, $\Phi_E = 0.35$, $T = 1.05$). (b) P-V diagram for fixed ($q_M = 0.028$, $\Phi_E = 0.35$) and varying T .

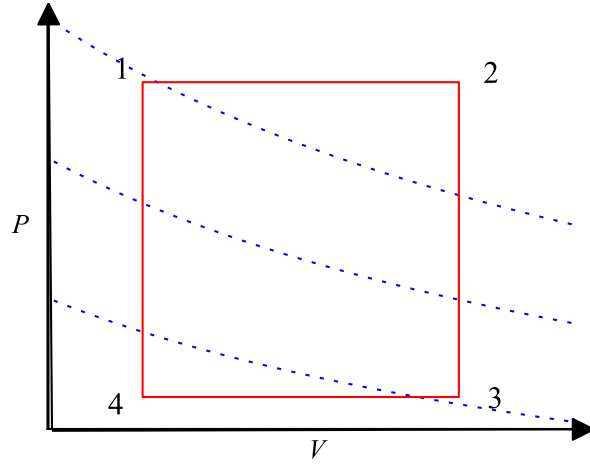


Figure 3. Our other engine.

$$\begin{aligned}
W &= \oint P dV \\
W_{total} &= W_{1 \rightarrow 2} + W_{3 \rightarrow 4} = P_1(V_2 - V_1) + P_4(V_4 - V_3) \\
W_{total} &= \frac{4}{3\sqrt{\pi}}(P_1 - P_4)(S_2^{\frac{3}{2}} - S_1^{\frac{3}{2}}).
\end{aligned} \tag{3.17}$$

The upper isobar will give the net inflow of heat which is Q_H , so one can write,

$$Q_H = \int_{T_1}^{T_2} C_P(P_1, T) dT. \tag{3.18}$$

In limit of high pressure we have,

$$T \sim \frac{2P\sqrt{S}}{\sqrt{\pi}}, \quad C_P \sim 2S = \frac{\pi T^2}{2P^2}, \tag{3.19}$$

which yields

$$Q_H \sim \frac{\pi}{6P_1^2}(T_2^3 - T_1^3) = \frac{4}{3\sqrt{\pi}}P_1(S_2^{\frac{3}{2}} - S_1^{\frac{3}{2}}). \tag{3.20}$$

Thus the efficiency is,

$$\eta = \frac{W}{Q_H} = 1 - \frac{P_4}{P_1}, \tag{3.21}$$

and also we have,

$$\eta = 1 - \frac{T_C}{T_H}. \tag{3.22}$$

4 Kerr Ads black hole solution

The Ads rotating black hole solution is given by the Kerr AdS metric [33]:

$$ds^2 = -\frac{\Delta}{\rho^2} \left[dt - \frac{a \sin^2 \theta}{\Xi} d\varphi \right]^2 + \frac{\rho^2}{\Delta} dr^2 + \frac{\rho^2}{s} d\theta^2 + s \frac{\sin^2 \theta}{\rho^2} \left[a dt - \frac{r^2 + a^2}{\Xi} d\varphi \right]^2, \tag{4.1}$$

where in $d = 4$

$$\rho^2 = r^2 + a^2 \csc^2 \theta, \quad \Xi = 1 - \frac{a^2}{\ell^2}, \quad \Delta = (r^2 + a^2)(1 + \frac{r^2}{\ell^2}) - 2mr, \quad s = 1 - \frac{a^2}{\ell^2} \csc^2 \theta. \tag{4.2}$$

The entropy, temperature and angular velocity is as follows,

$$S = \pi \frac{(r_+^2 + a^2)}{\Xi}, \quad T = \frac{r_+(1 + \frac{a^2}{\ell^2} + \frac{3r_+^2}{\ell^2} - \frac{a^2}{r_+^2})}{4\pi(r_+^2 + a^2)}, \quad \Omega_H = \frac{a\Xi}{r_+^2 + a^2}, \tag{4.3}$$

one can write the temperature as follows,

$$T = \frac{r_+}{4\pi} \left(-\frac{1}{r_+^2} + 8\pi P + \frac{2\pi}{S} \right). \tag{4.4}$$

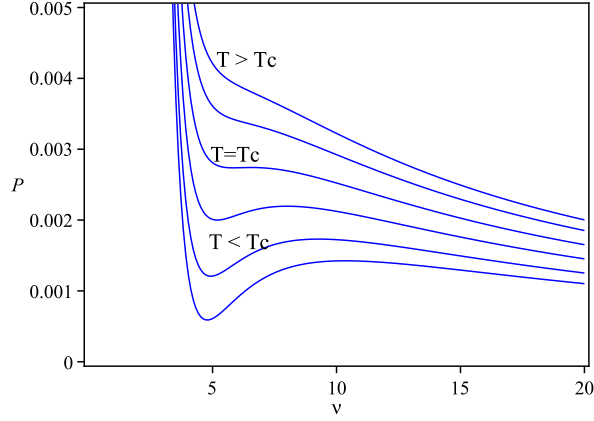


Figure 4. $P - v$ diagram for the rotating black hole. The isotherms in $P - v$ plane are depicted for a rotating black hole with $J = 1$.

The mass of black hole M and the angular momentum J are related to parameters m and a as follows,

$$M = \frac{m}{\Xi^2}, \quad J = \frac{am}{\Xi^2}. \quad (4.5)$$

The pressure identifies with $P = -\frac{\Lambda}{8\pi} = \frac{3}{8\pi} \frac{1}{\ell^2}$ and the thermodynamic volume is [15,17],

$$V = \frac{2\pi}{3} \frac{(r_+^2 + a^2)(2r_+^2 \ell^2 + a^2 \ell^2 - r_+^2 a^2)}{\ell^2 \Xi^2 r_+}, \quad (4.6)$$

one can write the above equation as follows,

$$V = \frac{2}{3} \left(\frac{S^2}{\pi r_+} + r_+ S \right). \quad (4.7)$$

The equation of state is written as by following equation,

$$P = \frac{T}{v} - \frac{1}{2\pi v^2} + \frac{48J^2}{\pi v^6}, \quad (4.8)$$

where

$$v = 2 \left(\frac{3V}{4\pi} \right)^{\frac{1}{3}} = 2r_+ + \frac{12J^2}{r_+(3r_+^2 + 8\pi r_+^4 P)}. \quad (4.9)$$

We illustrate $P - v$ diagram in Fig(4). The phase transition exists for this black hole too. Consequently, we can define heat engine for rotating black hole.

The heat capacity at constant pressure is,

$$C_P = \frac{2\pi r_+^2}{\Xi} \frac{\left(-\frac{1}{r_+^2} + 8\pi P + \frac{2\pi}{S}\right)}{\left(\frac{1}{r_+^2} + 8\pi P + \frac{2\pi}{S} - \frac{4\pi r_+^2}{S^2 \Xi}\right)}. \quad (4.10)$$

Here we use from limit $\frac{a^2}{r_+^2} \ll 1$, then one can obtain r_+ as follows,

$$r_+ \sim \sqrt{\frac{\Xi S}{\pi}}, \quad (4.11)$$

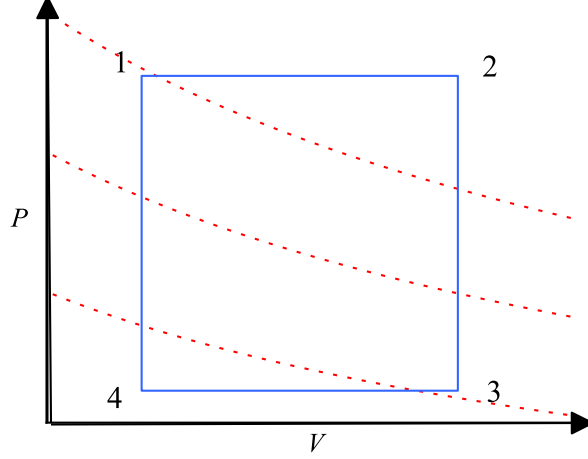


Figure 5. Our other engine.

and

$$V = \frac{2}{3} \sqrt{\frac{\Xi}{\pi}} S^{\frac{3}{2}} \left(\frac{1}{1 - \frac{a^2}{\ell^2}} + 1 \right), \quad (4.12)$$

as we know $\frac{a^2}{\ell^2} \ll 1$, so we have,

$$V = \frac{2}{3} \sqrt{\frac{\Xi}{\pi}} S^{\frac{3}{2}} \left(2 + \frac{a^2}{\ell^2} \right) \simeq \frac{4}{3} \sqrt{\frac{\Xi}{\pi}} S^{\frac{3}{2}}. \quad (4.13)$$

Consequently, the heat capacity at constant volume is $C_V = 0$. Thus we can define the following cycle for Kerr-AdS black hole.

The work done along the isobars is,

$$W = \frac{4}{3} \sqrt{\frac{\Xi}{\pi}} (S_2^{\frac{3}{2}} - S_1^{\frac{3}{2}}) (P_1 - P_4). \quad (4.14)$$

The inflow of heat determine of relation (25). As previous section, we use a limit which cycle is at high pressure, thus we have:

$$T \sim 2Pr_+, \quad (4.15)$$

where in limit $\frac{a^2}{r_+^2} \ll 1$ we have,

$$T \sim 2P \sqrt{\frac{\Xi S}{\pi}}. \quad (4.16)$$

The heat capacity at constant pressure in this limit is,

$$C_P \sim 2S \sim \frac{\pi}{2} \frac{T^2}{\Xi P^2}. \quad (4.17)$$

Consequently, one can obtain the following equation,

$$Q_H \sim \frac{\pi}{6\Xi P_1^2} (T_2^3 - T_1^3) = \frac{4}{3} \sqrt{\frac{\Xi}{\pi}} P_1 (S_2^{\frac{3}{2}} - S_1^{\frac{3}{2}}). \quad (4.18)$$

Therefore, the efficiency of this cycle is,

$$\eta = 1 - \frac{T_C}{T_H}. \quad (4.19)$$

5 Conclusion

In this paper we have studied thermodynamic cycle and heat engine for different black holes and we have compared to each other. By considering classical cycle for Dyonic black hole in constant electric potential and magnetic charge ensemble we have understood in case of static black holes, the thermodynamic volume V and entropy S are simply related to each other (this is key to the simplicity of one of the results concerning thermodynamic cycles). Therefore we can say adiabats and isochores are the same. By computation of the efficiency of this black hole we have seen the cycle has the maximum efficiency (Carnot efficiency) only at limit of high pressure. And far away of this limit it has efficiency less than Carnot efficiency. Then the next step, we have considered Kerr Ads black hole. In this case, the thermodynamic volume V and entropy S are independent, thus adiabats and isochores are not the same. But we have seen in limit $\frac{a^2}{r_+^2} \ll 1$ the volume and entropy are dependent and also adiabats and isochores are same. For that case computation of the efficiency will be simple. Also in this case, we can say that the efficiency has maximum value (which is same Carnot efficiency) at high pressure. Thus we note that, by assumption of high pressure limit for two black holes and limit $\frac{a^2}{r_+^2} \ll 1$ for Kerr Ads black hole, their behavior will be same. As we know the P-V critically may be not satisfied by any black hole. So, this lead us to consider special black holes for heat engine. Thus, this information help us to give correction to any black hole which is not satisfied by critical behavior. So we back this problem in future.

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